Abstract. Generation of non-reflecting boundary conditions for linear elastodynamics in anisotropic media is a challenging problem: commonly used perfectly matched layers (PML) [1] are unstable for some anisotropic media as has been proved in [2]. Here we formulate non-reflecting boundary conditions for the homogeneous anisotropic VTI axial-symmetric case on the basis of transparent boundary conditions (TBC) [3-6]. The resulting analytical formulas of TBC contain a non-local operator which we approximate numerically for efficient calculations. We provide numerical experiments demonstrated the high accuracy of the derived boundary conditions.

Introduction. Non-reflecting boundary conditions are required for numerical simulations of wave propagation in order to truncate the infinite physical space to a finite computational domain. One of the possible ways to construct these conditions is the approach of the transparent boundary conditions (TBC) [3-6] that can provide high accuracy and stability of computations with long simulation time. In frames of analytical TBC approach first introduced in [3] for the scalar wave equation and developed afterwards for lot of problems where it is possible (see [4-6] and references therein for the formulas for Maxwell system, linearized Euler equations and some others) the system of elastodynamics in anisotropic media has not been covered before.

Here we consider a particular but important case of rotationally symmetric elastic Vertical Transverse Isotropy media. Formulation of analytical TBC for this problem is still possible, although more technically sophisticated task comparing the problems solved before. Fortunately, in spite of the complicated intermediate formulas the final form of TBC is relatively simple. Another motivation of developing TBC for this problem is that the commonly used PML non-reflecting boundary conditions [1] are unstable for some elastic anisotropic media [2].

As always in TBC approach we are restricted by the case of the simple boundary – line here – and homogeneous media in the exterior domain though in the interior domain an arbitrary complex media is allowed. As well as for other problems (excluding the 1D wave equation) TBC are described with the operator that contains both local and non-local terms in time and space. The local term which is easily implemented numerically can be used for the problems when the moderate accuracy is required. To achieve the high accuracy we should include the non-local term which is approximated according to [4] for the efficient numerical discretization.

In this paper we describe the considered problem first, write out formulas of TBC then, discuss a way of the approximation of the boundary operator, report results of some numerical experiments, and give conclusions in the end.

Problem formulation. We consider a stress-velocity formulation of the elastodynamics equations for the rotationally symmetric case in anisotropic VTI media where the equations of motion are
\[ \rho \frac{\partial u_r}{\partial t} = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{oo}}{r} + \frac{\partial \sigma_{rz}}{\partial z}, \]
\[ \rho \frac{\partial u_z}{\partial t} = \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{\partial \sigma_{zz}}{\partial z}, \]

and the Hook’s law is
\[ \frac{\partial \sigma_{rr}}{\partial t} = A_{11} \frac{\partial u_r}{\partial r} + A_{12} \frac{u_r}{r} + A_{13} \frac{\partial u_z}{\partial z}, \]
\[ \frac{\partial \sigma_{oo}}{\partial t} = A_{12} \frac{\partial u_r}{\partial r} + A_{11} \frac{u_r}{r} + A_{13} \frac{\partial u_z}{\partial z}, \]
\[ \frac{\partial \sigma_{rz}}{\partial t} = A_{13} \frac{\partial u_r}{\partial r} + A_{11} \frac{u_r}{r} + A_{12} \frac{\partial u_z}{\partial z}, \]
\[ \frac{\partial \sigma_{zz}}{\partial t} = A_{44} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right). \]

Here \((u_r, u_z)\) and \(\sigma_{rr}, \sigma_{oo}, \sigma_{rz}, \sigma_{zz}\) are the components of the velocity vector and the stress tensor in \((r, z)\) coordinates respectively, \(\rho\) is the density, \(A_{11}, A_{12}, A_{13}, A_{33}, A_{44}\) are elastic parameters.

We consider a wave propagation in the infinite strip \(0 \leq r < \infty, Z_{\text{min}} \leq z \leq Z_{\text{max}}\) with simple boundary conditions on the top and bottom boundaries, \(z = Z_{\text{min}}, z = Z_{\text{max}}:\)
\[ u_z = 0, \sigma_{rz} = 0. \]

The computational domain is a rectangular \([0, R_r] \times [Z_{\text{min}}, Z_{\text{max}}]\) where on the right boundary \(r = R_r\) we will generate the transparent boundary conditions such that the outgoing waves leave the domain without reflection from the boundary. Parameters \(\rho, A_{11}, A_{12}, A_{13}, A_{33}, A_{44}\) are constant in space and time at \(r \geq R_r\).

**Transparent boundary conditions.** To derive TBC the procedure described in [4], [5], is applied, i.e. the conditions are formulated in a spectral space first (Fourier transformation along the boundary and Laplace transformation in time) and then transformed back into the physical space.

In the spectral space TBCs represent the relation between the unknowns and their normal derivatives which is derived by using the analytical solution of the exterior problem. Actually only two components of the unknowns vector of (1), (2) are included in the TBC operator: \(u_z, \sigma_{rz}\). In the spectral space TBCs have the form
\[ \frac{\partial}{\partial r} \begin{bmatrix} \hat{u}_r \\ \hat{\sigma}_{rz} \end{bmatrix} = \hat{P}(r, s, l) \begin{bmatrix} \hat{u}_r \\ \hat{\sigma}_{rz} \end{bmatrix}, \]
where “bar” and “hat” denote functions after Fourier transformation for \(z\) and Laplace transformation for \(t\), respectively. Variables \(l\) and \(s\) are correspondent dual variables. Matrix
\[ \hat{P} = \frac{(\chi_z - \chi_s)}{-2\sqrt{\xi t^4 + \eta \rho s^2 l^2 + \zeta^2 s^4}} \begin{pmatrix} a_o l^2 + a_s \rho s^2 & ib_{sl} \\ -ic_{sl} l^3 & -ic_{rs} l^3 - a_o l^2 - a_s \rho s^2 \end{pmatrix} + \frac{(\chi_z + \chi_s)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \]
scalar functions \(\chi_z, \chi_s\) are defined by
\[ \chi_s = \sqrt{\alpha l^2 + \beta ps^2 + \xi l^2 + \eta ps^2 l^2 + \zeta \rho^2 s^4} \times \frac{K_1 \left( r \sqrt{\alpha l^2 + \beta ps^2 + \xi l^2 + \eta ps^2 l^2 + \zeta \rho^2 s^4} \right)}{K_1 \left( r \sqrt{\alpha l^2 + \beta ps^2 + \xi l^2 + \eta ps^2 l^2 + \zeta \rho^2 s^4} \right)}. \]

Here \( K_1 \) is the Macdonald function of the first order (modified Bessel function of the second kind) and parameters \( a_0, a_2, b_1, c_1, c_{-1}, \alpha, \beta, \xi, \eta, \zeta \) are determined by the media properties:

\[
\begin{align*}
    a_0 &= \frac{A_{13} + A_{44}}{2A_{11}A_{44}}, & a_2 &= \frac{-A_{11} + A_{44}}{2A_{11}A_{44}}, & b_1 &= -\left( \frac{A_{13} + A_{44}}{A_{11}} \right), & c_1 &= \frac{(A_{11} + A_{44})}{A_{11}}, & c_{-1} &= -2A_{44}a_0; \\
    \alpha &= \frac{-A_{13} + 2A_{11}A_{44}A_{33}}{2A_{11}A_{44}}, & \beta &= \frac{A_{44}A_{11}A_{33}}{2A_{44}A_{44}}, & \xi &= \alpha^2 - \frac{A_{13}}{A_{11}}, & \eta &= 2\alpha\beta - \frac{A_{33} + A_{44}}{A_{11}}, & \zeta &= a_2^2.
\end{align*}
\]

In order to make inverse Laplace transform we represent the matrix \( \hat{P} \) in the following asymptotic form:

\[ \hat{P} = sQ_1 + \frac{1}{r}Q_0 + ilQ_2 + \hat{K}(s) + \frac{1}{s}Q_{-1}, \] (5)

where the matrices \( Q_0, Q_1, Q_2 \) depend on media parameters only; \( Q_{-1}, \hat{K}(s) \) depend on Fourier harmonic number \( l \), boundary position \( R_\Gamma \), and media parameters. Elements of \( \hat{K}(s) \) go to 0 as \( s \to +\infty \).

After inverse Laplace and Fourier transformations of (4) with \( \hat{P} \) represented as in (5) we obtain TBC formulas in the physical space:

\[ \begin{split}
    &Q_1 \frac{\partial}{\partial t} \left[ \frac{u_r}{\sigma_{rz}} \right] - \frac{\partial}{\partial r} \left[ \frac{u_r}{\sigma_{rz}} \right] + \frac{1}{r}Q_0 \left[ \frac{u_r}{\sigma_{rz}} \right] + Q_2 \frac{\partial}{\partial \tau} \left[ \frac{u_r}{\sigma_{rz}} \right] \\
    &+ F^{-1}K * F \left[ \frac{u_r}{\sigma_{rz}} \right] + F^{-1}Q_1F \left[ \frac{u_r}{\sigma_{rz}} \right] d\tau = 0.
\end{split} \] (6)

Here \( F \) is the Fourier transform operator; \( K(t) \) is the inverse Laplace transform of \( \hat{K}(s) \); asterisk * denotes convolution with respect to time.

**Approximation of the TBC.** Boundary conditions in (6) are exact but computationally expensive due to the non-local terms, first of all because of the time convolution that depends on the whole history of the solution on the boundary. To make TBC computationally efficient we approximate (6) in the similar way as it is done in [4], [5].

A simple and computationally cheap approximation of (6) is achieved by omitting the term with convolution. Although the accuracy of such boundary condition should not be high it can be helpful for some particular problems.

For more accurate boundary conditions the kernel of the convolution \( K(t) \) is approximated via the sum of the exponentials. As the convolution with an exponential can be calculated by recurrence formulas the implementation of the non-local term of the approximated TBC is localized in time and requires reasonable computational resources. To construct the approximation we use the algorithm suggested in [4] where the rational Padé-Chebyshev approximation of the kernel function \( \hat{K}(s) \) from (5) provides us with
the approximation by a sum of the exponentials in time domain. Accuracy of such TBC is determined by the accuracy of the approximation and can be improved by using the larger number of exponentials.

**Numerical experiments.** We implement TBC (6) into the Virieux scheme [7] on the staggered grid for the system (1), (2). On the boundary \( r = R_r \) of the computational domain we update the components \( u_r \) and \( \sigma_{rz} \) using the TBC equations (6) whereas for the other components we use the main equations (1), (2).

In the numerical experiments we consider the computational domain with \( R_r = 0.3 \) and \( Z_{\text{max}} - Z_{\text{min}} = 2.5 \) and anisotropic media described by the Thompsen parameters [8]: \( V_p = 4.449 \), \( V_s = 2.585 \), \( \rho = 2.57 \), \( \delta = 0.565 \), \( \varepsilon = 0.0455 \), \( \gamma = 0.046 \), for which PML conditions are unstable (the formulas for transforming these parameters into the set of \( A_{11}, A_{12}, A_{13}, A_{33}, A_{44} \) can be found in [8]).

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**Fig. 1.** Relative accuracy of TBC for two grids vs. number of the Fourier harmonic. Left: non-local TBC; right: local boundary conditions.

Here we describe one of the main of our usual tests for validation TBC: separate one-dimensional problems for each parameter \( l \) after Fourier transformation with respect to \( z \). We generate the wave process that contains both quasi P- and S-waves with the approximately equaled amplitudes by a source term for the all stress components located in \( r = 0.25 \). The source is described by the Ricker pulse \( (1 - 2\pi^2 t_0^2 (t - d)^2) e^{-\pi^2 t_0^2 (1 - d)^2} \) of time with the central frequency \( \nu_0 = 20 \) and delay \( d = 2/\nu_0 \). For the source dependence on \( z \) we use the correspondent Fourier harmonic. The simulation time is \( T = 0.65 \). We use two grids: the coarse one with 256 cells in \( r \) direction and the fine one with 512 cells. We estimate the accuracy of the numerical solutions comparing with the reference solution in a fixed spatial point near the boundary \( r = R_r \) in \( C \)-norm on the interval \([0, T]\). The reference solution is obtained on a very fine grid in the domain extended in \( r \) direction so that the reflections from the right boundary do not reach the domain of interest during the considered simulation time. In Fig. 1 we present calculated relative accuracy for the tests for the first fifteen harmonics \((l = 0, 1, \ldots, 14)\). As expected the accuracy of the local boundary conditions is low and doesn’t decrease with the mesh refinement (see graph at the right). On the contrary the accuracy of our TBC with the non-local terms is high (the residual is less than \( 2 \cdot 10^{-3} \)
for the fine grid) so the finite-difference solution follows the 2nd order of accuracy of the Virieux scheme (see graph at the left).

**Conclusions.** The analytic transparent boundary conditions for the elastic anisotropic VTI media in the axial-symmetric case are formulated. An efficient approximation and implementation of the TBC operator into the Virieux scheme are developed. Numerical experiments demonstrate the expected high accuracy and stability of the proposed conditions.

**References**

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